

Inverse counting statistics based on generalized factorial cumulants

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Abstract. We propose a method to reconstruct characteristic features of an unknown stochastic system from the long-time full counting statistics of some of the system's transitions that are monitored by a detector. The full counting statistics is conveniently parametrized by so-called generalized factorial cumulants. Taking only a few of them as input information is sufficient to reconstruct important features such as the lower bound of the system dimension and the full spectrum of relaxation rates. The use of generalized factorial cumulants reveals system dimensions and rates that are hidden for ordinary cumulants. We illustrate the inverse counting-statistics procedure for two model systems: a single-level quantum dot in a Zeeman field and a single-electron box subjected to sequential and Andreev tunneling.

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1. Introduction

A stochastic system is characterized by the rates of the stochastic transitions between the possible states. The dynamics of the stochastic system may be probed by a detector that is sensitive to one or several (but, in general, not all) of the possible transitions. In biological physics, stochastic transitions like the steps of motor proteins [1, 2], intramolecular conformational changes [3–5], and enzymatic turnovers generating fluorescent products [6, 7] have been studied. Detectors are optical tweezers [1, 8], atomic force, or fluorescence microscopes [8, 9]. In mesoscopic physics, discrete charge-transfer events [10, 11] through single- [12–14] and multi-level [15] quantum dots, interferometers [16], superconducting [17–23] and feedback-controlled systems [24–26] have been studied. Charge transfers can be detected by a quantum point contact [27–35], a single-electron transistor [36–40], optical [41], or interferometric detectors [42].

Counting the number N of transitions within a time interval $[0; t]$ repeatedly many times yields the probability distribution $P_N(t)$, referred to as *full counting statistics*. For a well-characterized system, the states, the rates between them, and the coupling to the detector are known. It is, then, straightforward to compute the full counting statistics and compare it with experimentally measured data. Suppose, however, that the underlying model for a stochastic system is unclear and the only information available is the counting statistics measured by the detector. It is, then, desirable to have a systematic approach to distill out of the measured counting statistics the relevant information for reconstructing properties of the underlying model, including basic properties such as the number of states of the stochastic system. Such an approach can be dubbed *inverse counting statistics* [43].

What are the properties of the stochastic system that one may hope to reconstruct by inverse counting statistics? First of all, there is the number M of possible system states. Second, the stochastic system is characterized by the spectrum of relaxation rates with which it relaxes back to its (equilibrium or nonequilibrium) steady state after being disturbed externally [44–49]. Of course, inverse counting statistics cannot distinguish between different stochastic systems that are equivalent in the sense that they produce the same counting statistics, even if the stochastic systems possess a different numbers of states. Therefore, inverse counting statistics can at most deliver the *minimal* number of system states necessary that is compatible with the observed counting statistics.

To make the inverse counting statistics a powerful and practical tool, one should keep the circumstances for the acquisition of the input data as transparent and as simple as possible. Therefore, we restrict ourselves to the following scenario. First, only steady-state counting statistics is considered, i.e., we assume that the system has already relaxed before counting starts. This excludes studying transient behavior after a perturbation of the system. The latter would, on the one hand, offer a direct access to some relaxation rate of the system [50–52]. On the other hand, the determination of the full spectrum of relaxation rates would require the knowledge of how to perturb the system in order to probe a specific relaxation rate.

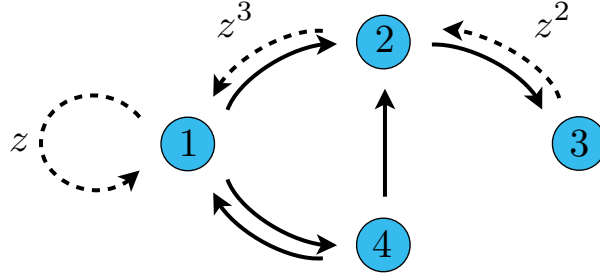


Figure 1. Stochastic system with $M = 4$ states. Transitions are indicated by arrows. Dashed arrows indicate transitions counted by the detector with counting factors z^k .

Second, we concentrate on the limit of long measuring-time intervals $[0; t]$, for which the system dynamics is dominated by the slowest relaxation rate only. Nevertheless, the procedure of inverse counting statistics yields the full spectrum of all relaxation rates, as explained below.

Given a measured distribution $P_N(t)$, what are the input data for the inverse counting statistics? In [43], it was suggested to use the (long-time) cumulants of the distribution function. Here, we propose to use generalized factorial cumulants [53, 54] instead. The advantage of the latter is that they depend on an arbitrarily chosen parameter s . The outcome of the inverse counting statistics (such as the number of system states or the spectrum of relaxation rates) should, however, not depend on this parameter s . Therefore, the s -independence of the results defines a powerful consistency criterion. Furthermore, as we will see in section 5, there are special cases in which part of the relaxation-rate spectrum is not accessible by inverse counting statistics with ordinary cumulants but is detectable by using generalized factorial cumulants with properly chosen parameter s . As another difference to [43], where the detector was assumed to be sensitive to only a single transition between two specific states increasing the detector counter just by one, we allow for a more general system-detector coupling, taking into account arbitrarily many transitions between arbitrarily many states increasing the detector counter by an arbitrary amount. Therefore, our inverse counting procedure does not only test compatibility with the number M of system states but also with a parameter m that depends on the system-detector coupling as explained below.

The paper is organized as follows. In section 2, we give a short introduction how full counting statistics, especially generalized factorial cumulants, are calculated by means of a Markovian master equation. Subsequently, in section 3, we explain the general procedure of inverse counting statistics. This procedure is, then, illustrated in sections 4 and 5 for two model systems: a single-level quantum dot in a Zeeman field and a single-electron box subjected to sequential and Andreev tunneling.

2. Full counting statistics for stochastic systems

We consider stochastic systems as represented in figure 1. There are M system states, labeled by χ . Systems with coherent superposition of different χ are not taken into account. Arrows indicate transitions from state χ to state χ' with a transition rate $\Gamma_{\chi'\chi}$. The counting factors z^k with $k = 1, 2, \dots$ next to the dashed arrows specify the coupling of the system to the detector: if the system undergoes such a transition, the number N counted by the detector increases by k . There may be also processes with rate $\Gamma_{\chi\chi}$ monitored by the detector in which the system ends up in the same state χ as it started, which is, e.g., the case for cotunneling through a magnetic atom [55].

We assume that the duration of all the transition processes is shorter than the time resolution of the detector. In this case, the system's dynamics can be described by a Markovian master equation, and non-Markovian effects [56–58] do not need to be taken into account.

Without the detector, the master equation for the probabilities $p^\chi(t)$ to find the system in state χ takes the form

$$\dot{p}^\chi(t) = \sum_{\chi'} [\Gamma_{\chi\chi'} p^{\chi'}(t) - \Gamma_{\chi'\chi} p^\chi(t)]. \quad (1)$$

The first term of the sum describes transitions into state χ , the second term transitions out of state χ . To account for the detector, we need to replace this master equation for $p^\chi(t)$ by an N -resolved one for $p_N^\chi(t)$, where N is the number of detector counts in the time interval $[0; t]$. For this, we introduce the coupling constants $d_{\chi\chi'}^k$ which is 1 if the detector count is increased by k for the transition from χ' to χ , and 0 otherwise. We obtain

$$\dot{p}_N^\chi(t) = \sum_{\chi'} \sum_k d_{\chi\chi'}^k \Gamma_{\chi\chi'} p_{N-k}^{\chi'}(t) - \sum_{\chi'} \Gamma_{\chi'\chi} p_N^\chi(t). \quad (2)$$

Solving for the N -resolved probabilities yields the counting statistics of the detector via $P_N(t) = \sum_\chi p_N^\chi(t)$.

We now turn to the question of how to extract from a given distribution $P_N(t)$ the information that serves as an input for the inverse counting statistics. One possibility would be to use the (long-time) cumulants of the distribution function, as suggested in [43]. Here, we propose, as an alternative, to employ generalized factorial cumulants [53] in the long-time limit. Generalized factorial cumulants are derived from the generating function

$$\mathcal{M}_s(z, t) := \sum_{N=0}^{\infty} (z + s)^N P_N(t), \quad (3)$$

by evaluating the derivatives

$$C_{s,k}(t) := \left. \frac{\partial^k \ln \mathcal{M}_s(z, t)}{\partial z^k} \right|_{z=0}. \quad (4)$$

The special case $s = 1$ recovers the factorial cumulants recently introduced in the context of mesoscopic transport [59, 60]. They are called *factorial* because the corresponding moments $\partial_z^k \mathcal{M}_1(z, t)|_{z=0} = \langle N^{(k)} \rangle$ are expectation values of the *factorial* power $N^{(k)} := N(N-1)\dots(N-k+1)$ instead of the ordinary power N^k which defined ordinary moments and cumulants. *Generalized* factorial cumulants depend on an extra parameter s that describes a shift of the complex variable z by the amount $s-1$ along the real axis in the complex plane. In contrast to previous works [53, 54], we define the generating function without a normalization factor $1/\mathcal{M}_s(0, t)$. Such a z -independent normalization factor would not influence the cumulants of order $k > 0$, but it would set, per definition, $C_{s,0} = 1$. Without this normalization factor, $C_{s,0}(t) = \sum_N s^N P_N(t)$ contains non-trivial information that can be used for the inverse counting statistics. Both factorial ($s = 1$) and generalized factorial cumulants ($s \neq 1$) have been utilized to detect correlations in charge-transfer statistics [53, 54, 59–61].

In the context of inverse counting statistics, the parameter s will play a very important role in two respects. First, since the underlying stochastic model for a measured distribution $P_N(t)$ is independent of s , the outcome of the inverse counting statistics must also be independent of the parameter s . Therefore, the required s -independence of the obtained results defines a criterion for the compatibility of the measured data with the assumed underlying model. Second, as we will see below, there are special cases in which the inverse counting statistics with factorial cumulants ($s = 1$) would indicate compatibility with a too small stochastic system and only generalized factorial cumulants ($s \neq 1$) reveal the higher dimension of the underlying stochastic model.

To calculate the generalized factorial cumulants for a given stochastic system, it is convenient to first perform a z -transform of the N -resolved master equation equation (2), i.e., multiply with z^N and then sum over N . If we combine the z -transformed N -resolved probabilities $p_z^\chi = \sum_N z^N p_N^\chi$ of the different states χ in a vector $\mathbf{p}_z(t)$, the z -transformed master equation can be written in the form

$$\dot{\mathbf{p}}_z(t) = \mathbf{W}_z \mathbf{p}_z(t), \quad (5)$$

with matrix elements

$$(\mathbf{W}_z)_{\chi\chi'} = \sum_k z^k d_{\chi\chi'}^k \Gamma_{\chi\chi'} - \delta_{\chi\chi'} \sum_{\chi''} \Gamma_{\chi''\chi'}. \quad (6)$$

Two examples for \mathbf{W}_z are given in equations (14) and (18). For $z = 1$, equation (5) is nothing but the master equation equation (1).

The solution of equation (5) is $\mathbf{p}_z(t) = \exp(\mathbf{W}_z t) \mathbf{p}_z(0)$. Since $p_N^\chi(0) \sim \delta_{N,0}$, the initial vector $\mathbf{p}_z(0)$ is independent of z and describes the initial probability distribution. The matrix exponential $\exp(\mathbf{W}_z t) = \sum_{j=1}^M \exp[\lambda_j(z)t] \mathbf{r}_{j,z} \otimes \mathbf{l}_{j,z}^T$ can be expressed in terms of the eigenvalue spectrum $\{\lambda_j(z)\}$ of \mathbf{W}_z by making use of the decomposition into the left and right eigenvector $\mathbf{l}_{j,z}$ and $\mathbf{r}_{j,z}$ with normalization $\mathbf{l}_{j,z}^T \cdot \mathbf{r}_{j',z} = \delta_{jj'}$. For an arbitrary initial distribution $\mathbf{p}_z(0)$ of the system, the systems relaxes exponentially

in time to its steady state, governed by the eigenvalues $\lambda_j(z)$. The eigenvalues $\lambda_j(z)$ at $z = 1$ are the system's relaxation rates mentioned in the introduction. They are either real or they appear as complex-conjugated pairs. In the following, we assume that electron counting starts only after the system has reached its steady state, i.e., $\mathbf{p}_z(0)$ is the stationary probability distribution, determined by $\mathbf{W}_1 \mathbf{p}_z(0) = 0$ and $\mathbf{e}^T \cdot \mathbf{p}_z(0) = 1$, where we defined $\mathbf{e}^T = (1, \dots, 1)$ to sum over all states χ in $\mathbf{p}_z(0)$.

Finally, taking into account that the generating function can be written as $\mathcal{M}_s(z, t) = \mathbf{e}^T \cdot \mathbf{p}_{z+s}(t)$, we obtain

$$C_{s,k}(t) = \frac{\partial^k}{\partial z^k} \ln \left[\sum_{j=1}^M (\mathbf{e}^T \cdot \mathbf{r}_{j,z}) (\mathbf{l}_{j,z} \cdot \mathbf{p}_z(0)) e^{\lambda_j(z)t} \right]_{z=s}. \quad (7)$$

The summation over j complicates the time dependence of the generalized factorial cumulants. In the long-time limit, however, the above expression becomes considerably simpler since the exponential factors suppress all terms of the sum except the ones with the largest real part $\text{Re}[\lambda_j(z)]$ for $z = s$. For $z = 1$ and systems with a unique stationary state, the dominant eigenvalue is 0, i.e., all other eigenvalues have a negative real part. Around $z = 1$, the dominant eigenvalue, denoted by $\lambda_{\max}(z)$, remains real and the limit

$$c_{s,k} := \lim_{t \rightarrow \infty} \frac{C_{s,k}(t)}{t} = \left. \frac{\partial^k \lambda_{\max}(z)}{\partial z^k} \right|_{z=s}, \quad (8)$$

provides well-defined constants, referred to as scaled long-time (generalized factorial) cumulants. These scaled long-time cumulants define the input information for the inverse counting statistics. If s is chosen very negative, it may happen that the dominant eigenvalues are given by a complex-conjugated pair. In this case, the limit is not well defined and the inverse-counting-statistics procedure derived below cannot be applied.

3. Inverse Counting Statistics

In the previous section, we have shown how to calculate for a given stochastic model defined by the matrix \mathbf{W}_z (referred to as the generator of the stochastic system) the long-time (generalized factorial) cumulants. Inverse counting statistics deals with the opposite problem: how much can we learn about the stochastic system if only a few numbers, namely the experimentally determined values of the scaled long-time cumulants (up to some order), are given? To be more specific, we aim at the following properties of the stochastic system. First, a very important feature of the stochastic system is the dimension M of \mathbf{W}_z , i.e., the number of participating states in the stochastic process. Furthermore, the coupling to the detector is described by powers of z attached to some matrix elements of the generator. As a consequence, the characteristic polynomial $\det(\lambda \mathbf{1} - \mathbf{W}_z)$ is of order m in z . Thus, as a second feature, we identify the m . We will show below that the values of the first $(m+1)M$ scaled long-time cumulants are enough to check compatibility with a stochastic system of dimension M and the order m characterizing the coupling to the detector.

But, with inverse counting statistics, we can get much more. From the $(m+1)M$ input parameters $c_{k,s}$ it is possible to determine the full spectrum of \mathbf{W}_z , i.e., the full z -dependence of the eigenvalues $\lambda_j(z)$. To appreciate how remarkable this statement is, let us remind that the input parameters are only a finite $[(m+1)M]$ number of derivatives of only one eigenvalue λ_{\max} at only one value of z , namely the arbitrarily chosen s . From this rather restricted amount of information, we aim at reconstructing also the other eigenvalues different from λ_{\max} at all values of z different from s . How is this possible and how does it work in practice?

To answer this question, we observe that the characteristic function of the generator \mathbf{W}_z ,

$$\chi(z, \lambda) = \det(\lambda \mathbf{1} - \mathbf{W}_z) = \prod_{j=1}^M (\lambda - \lambda_j(z)) \quad (9)$$

is a polynomial both in λ (of order M) and in z (of order m). The eigenvalues $\lambda_j(z)$, i.e., the roots of the characteristic function, $\chi(z, \lambda_j(z)) = 0$, are, in general, nonanalytic functions in z . The characteristic function $\chi(z, \lambda)$ itself, however, is a polynomial in z and can, therefore, be written in the form

$$\chi(z, \lambda) = \lambda^M + \sum_{\mu=0}^m \sum_{\nu=0}^{M-1} a_{\mu\nu} (z-s)^\mu \lambda^\nu, \quad (10)$$

where s is the arbitrarily chosen parameter of the generalized factorial cumulants. As a consequence, the (s -independent) characteristic function is fully determined by the $(m+1)M$ real (and s -dependent) coefficients $a_{\mu\nu}$. This fixes all z -dependent (but s -independent) eigenvalues $\lambda_j(z)$ of the generator \mathbf{W}_z . For this reason, $(m+1)M$ input parameters are enough to fully determine the spectrum of \mathbf{W}_z .

Suppose that M and m are already known (we will discuss below how this is done with the help of inverse counting statistics). How do we get the spectrum of \mathbf{W}_z ? As input parameters we use the scaled generalized factorial cumulants $c_{s,k}$ for $k = 0, \dots, (m+1)M - 1$ in the long-time limit. To determine the coefficients $a_{\mu\nu}$, we perform $l = 0, \dots, (m+1)M - 1$ times a derivative of $\chi(z, \lambda(z)) \equiv 0$ with respect to z and set $z = s$ afterwards. For technical reasons, it is convenient to divide the resulting equation by $l!$. Then, we arrive at the set of linear equations

$$0 = A_{l,0M} + \sum_{\mu=0}^m \sum_{\nu=0}^{M-1} A_{l,\mu\nu} a_{\mu\nu}, \quad (11)$$

for $l = 0, \dots, (m+1)M - 1$. The coefficients $A_{l,\mu\nu}$, defined for nonnegative l, μ , and ν , are given by

$$A_{l,\mu\nu} = \begin{cases} 0 & \text{for } l < \mu \\ 1 & \text{for } l = \mu, \nu = 0 \\ 0 & \text{for } l > \mu, \nu = 0 \end{cases} \quad (12)$$

and otherwise (i.e., $l \geq \mu$ together with $\nu \geq 1$) by

$$A_{l,\mu\nu} = \sum_{\alpha_1 + \dots + \alpha_\nu = l - \mu} \frac{c_{s,\alpha_1}}{\alpha_1!} \cdot \frac{c_{s,\alpha_2}}{\alpha_2!} \dots \frac{c_{s,\alpha_\nu}}{\alpha_\nu!}. \quad (13)$$

Obviously, $A_{l,\mu\nu}$ depends on l and μ only via the difference $l - \mu$ (this was the reason of dividing by $l!$). The multiple sum over the α 's is constrained by $\alpha_1 + \dots + \alpha_\nu = l - \mu$. An alternative expression for the $A_{l,\mu\nu}$ can be found in [Appendix A](#).

To be explicit, let us write down all the terms that are relevant for the case $M = 3$ and $m = 2$. We need ν up to 2 and l up to 8. For $\nu = 0$ we get $A_{l,\mu 0} = \delta_{l\mu}$, for $\nu = 1$ we have $A_{l,\mu 1} = c_{s,l-\mu}/(l - \mu)!$, and for $\nu = 2$ we find

$$\begin{aligned} A_{\mu,\mu 2} &= c_{s,0}^2, \\ A_{\mu+1,\mu 2} &= 2c_{s,0}c_{s,1}, \\ A_{\mu+2,\mu 2} &= c_{s,0}c_{s,2} + c_{s,1}^2, \\ A_{\mu+3,\mu 2} &= \frac{c_{s,0}c_{s,3}}{3} + c_{s,1}c_{s,2}, \\ A_{\mu+4,\mu 2} &= \frac{c_{s,0}c_{s,4}}{12} + \frac{c_{s,1}c_{s,3}}{3} + \frac{c_{s,2}c_{s,2}}{4}, \\ A_{\mu+5,\mu 2} &= \frac{c_{s,0}c_{s,5}}{60} + \frac{c_{s,1}c_{s,4}}{12} + \frac{c_{s,2}c_{s,3}}{6}, \\ A_{\mu+6,\mu 2} &= \frac{c_{s,0}c_{s,6}}{360} + \frac{c_{s,1}c_{s,5}}{60} + \frac{c_{s,2}c_{s,4}}{24} + \frac{c_{s,3}c_{s,3}}{36}, \\ A_{\mu+7,\mu 2} &= \frac{c_{s,0}c_{s,7}}{2520} + \frac{c_{s,1}c_{s,6}}{360} + \frac{c_{s,2}c_{s,5}}{120} + \frac{c_{s,3}c_{s,4}}{72}, \\ A_{\mu+8,\mu 2} &= \frac{c_{s,0}c_{s,8}}{20160} + \frac{c_{s,1}c_{s,7}}{2520} + \frac{c_{s,2}c_{s,6}}{720} + \frac{c_{s,3}c_{s,5}}{360} + \frac{c_{s,4}c_{s,4}}{576}. \end{aligned}$$

To obtain the full spectrum $\{\lambda_j(z)\}$, one first solves the set of linear equations equation (11) for $a_{\mu\nu}$. Second, the result for $a_{\mu\nu}$ is inserted into equation (10) to get the characteristic function. Finally, the zeros of the characteristic function are determined. It all works because the characteristic function is fully determined by a finite number of coefficients $a_{\mu\nu}$ only.

It is important to remark that, in general, there is no guarantee that the set of linear equations equation (11) provides a unique solution. There may be special situations (we will discuss such a case below) in which the generator \mathbf{W}_z is separable in the sense that its characteristic function can be written as a product of two polynomials, the first one of order m' and M' in z and λ , the second one of order $m - m'$ and $M - M'$. Of course, it is trivial that the characteristic function can always be written as a product of two polynomials in λ , but separability requires that, in addition, the two polynomials in λ are polynomials in z as well. For separable generators, inverse counting statistics has only access to the part of the spectrum to which the eigenvalue $\lambda_{\max}(z)$ with the largest real part belongs, i.e., the effective problem has reduced dimensions M' and m' .

The remaining question to be answered is how to determine the dimensions M and m of the stochastic system. In the spirit of [43], one may suggest that after having determined $\lambda_{\max}(z)$ from the first $(m+1)M$ scaled long-time cumulants one can calculate

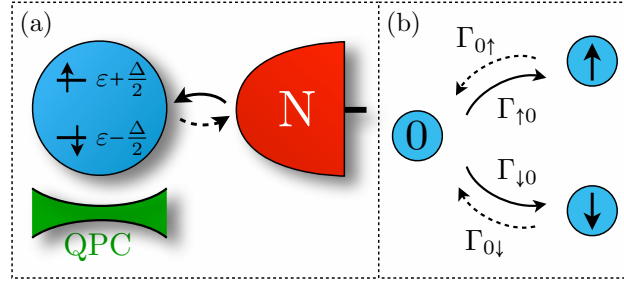


Figure 2. (Color online) (a) A single-level quantum dot subjected to a Zeeman field is tunnel coupled to one normal lead. (b) Sketch of the states and transition rates. Dashed arrows indicate the counted transitions.

the scaled cumulants of higher order and compare them with experimentally measured values. If M and m were chosen correctly, then one expects a coincidence of calculated and measured values. This has, however, two downsides. First, measuring cumulants of increasingly higher order may become more and more difficult. Second, comparing just numbers to establish a consistency criterion may be of only limited significance in view of experimentally unavoidable noise.

With the use of generalized factorial cumulant, however, we can do much better. Remember that the parameter s was chosen arbitrarily and the result, i.e., the spectrum of the generator, must be independent of this parameter s . Therefore, we can use the very same measured time trace of the detector to determine scaled cumulants for different values of s , run the inverse-counting-statistics procedure for each s . If M and m were chosen correctly, then the full z -dependence of the full spectrum should be independent of the choice of s . This establishes a much stronger consistency check than comparing just a few numbers.

In the following sections 4 and 5, we illustrate the inverse-counting statistics procedure for two model systems: a single-level quantum dot in a Zeeman field and a single-electron box subjected to sequential and Andreev tunneling.

4. Single-level quantum dot in a Zeeman field

The first model system is depicted in figure 2(a). A single-level quantum dot is weakly tunnel coupled to one normal-state metallic lead N and subjected to a magnetic field. The orbital level ϵ measured relative to lead's electrochemical potential is splitted by the Zeeman energy Δ into $\epsilon_\sigma = \epsilon \pm \Delta/2$. The positive (negative) sign applies to a spin $\sigma = \uparrow (\downarrow)$ electron on the quantum dot. The current through a nearby quantum point contact is sensitive to the dot charge which allows to monitor the charge transfer between dot and lead as function of time. The empty dot 0 can be occupied by a spin σ electron with the sequential tunneling rate $\Gamma_{\sigma 0} = \Gamma f(\epsilon_\sigma)$ given by Fermi's golden rule. The reverse transition occurs with the rate $\Gamma_{0\sigma} = \Gamma [1 - f(\epsilon_\sigma)]$, where $f(\epsilon_\sigma) = [1 + \exp(\epsilon_\sigma/k_B T)]^{-1}$ is the Fermi function. The temperature T is as large that

both $\Gamma_{\sigma 0}$ and $\Gamma_{0\sigma}$ are nonvanishing, but transitions to higher charge states are negligible due to charging energy.

The stochastic system is depicted in figure 2(b). Its generator is given by

$$\mathbf{W}_z = \begin{pmatrix} -\Gamma_{\uparrow 0} - \Gamma_{\downarrow 0} & z\Gamma_{0\uparrow} & z\Gamma_{0\downarrow} \\ \Gamma_{\uparrow 0} & -\Gamma_{0\uparrow} & 0 \\ \Gamma_{\downarrow 0} & 0 & -\Gamma_{0\downarrow} \end{pmatrix}. \quad (14)$$

The characteristic function is a polynomial of order $M = 3$ in λ and of order $m = 1$ in z .

The case of vanishing magnetic field $\Delta = 0$, however, is special because of spin degeneracy $\varepsilon_{\uparrow} = \varepsilon_{\downarrow}$, and only two different rates $\Gamma_{\uparrow 0} = \Gamma_{\downarrow 0} = \Gamma_{10}$ and $\Gamma_{0\uparrow} = \Gamma_{0\downarrow} = \Gamma_{01}$ appear. As a consequence, the characteristic function becomes separable,

$$\chi(z, \lambda) = \chi_{1,2}(z, \lambda) \cdot \chi_3(\lambda) \quad (15)$$

$$\chi_{1,2}(z, \lambda) = \lambda^2 + (\Gamma_{01} + 2\Gamma_{10})\lambda - 2(1 - z)\Gamma_{01}\Gamma_{10} \quad (16)$$

$$\chi_3(\lambda) = \lambda + \Gamma_{01}. \quad (17)$$

The first factor is a polynomial of order $M = 2$ in λ and of order $m = 1$ in z , while the second factor is of order $M = 1$ and independent of z . Due to the z -independency, the second factor does not influence counting statistics and thus can not be detected anymore. The electron transfer can be completely described by a spinless orbital which is occupied with rate $2\Gamma_{10}$ and emptied with rate Γ_{01} . Note that a separable characteristic function does not always separate in a z -dependent and z -independent factor, an example will be given in section 5.

4.1. Nonvanishing magnetic field

We start with discussing the case of nonvanishing magnetic field for which we choose $\varepsilon = -k_B T$ and $\Delta = k_B T/2$. The input information for the inverse counting statistics (for $M = 3$ and $m = 1$) is given by the scaled long-time generalized factorial cumulants from order 0 up to order $(m + 1)M - 1 = 5$. Since in experiments, the measurement time is always finite, we do not take as input parameters the exact long-time scaled cumulants of the defined model but calculate, instead, the scaled cumulants at some large but finite time. Hence, the scaled cumulants are close but not identical to the exact values in the long-time limit. For the exact long-time scaled cumulants, a value for the dimension M that was assumed to be too large, can be immediately identified by trying to solve the system of linear equations (11). Then, no unique solution for every $a_{\mu\nu}$ is obtained, i.e., the linear equations are not independent from each other. However, in the following, we stick to the experimental situation that, due to the finite measuring time or due to experimental noise, a unique solution for the system of linear equations (11) is even obtained if M is assumed too large.

First, we determine the eigenvalue spectrum from the inverse counting statistics performed at $s = 1$ and assuming the correct values $M = 3$ and $m = 1$. As

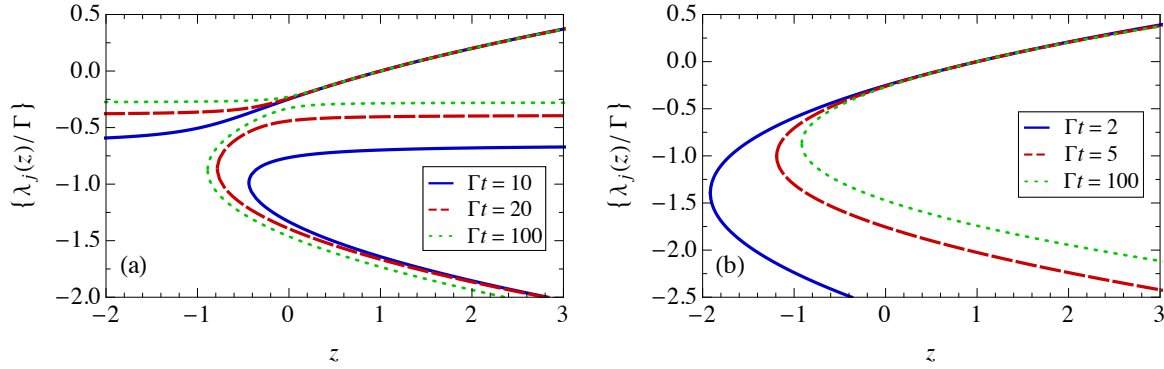


Figure 3. (Color online) Eigenvalue spectrum obtained via inverse counting statistics for a single-level quantum dot $\varepsilon = -k_B T$ (a) inside a magnetic field $\Delta = k_B T / 2$ and (b) without magnetic field. At the time $\Gamma t = 100$ no difference to the exact eigenvalues, obtained from equation (14), can be recognized anymore. Eigenvalues with a finite imaginary part (occurring for very negative z) are not depicted.

input parameter we take the calculated scaled generalized factorial cumulants at times $\Gamma t = 10, 20, 100$. The result is shown in figure 3(a). For $\Gamma t = 100$, no difference to the exact results can be recognized anymore.

Next, we demonstrate the consistency check for the dimensions M and m . For this, we check the required s -independence of the eigenvalue spectrum. In the following, we always use as input information the calculated scaled cumulants at $\Gamma t = 6000$ to ensure convergence to the asymptotic long-time behavior not only for $s = 1$ but also for other s .

To show simultaneously both the z - and the s -dependence of the eigenvalues, we plot in the following figures the contour lines for different selected values of λ (in units of Γ). Horizontal contour lines indicate that the eigenvalues are independent of s , i.e., the assumed dimensions M and m are compatible with the input data. In figure 4(a) and (b), we show the result for the choice $M = 2$ and $m = 1$. Since the dimension M of the stochastic systems is taken too small, the resulting eigenvalues display a strong s -dependence. However, if we take the proper values $M = 3$ and $m = 1$, see figure 4(c), (d), (e), we get s -independent results. The order $M = 3$ and $m = 1$ are lower bounds for the system's dimensions.

4.2. Vanishing magnetic field

We now turn to the case of vanishing magnetic field $\Delta = 0$, for which the characteristic function is separable, i.e., the characteristic function is a product of two polynomials, one of order $M = 2$ and $m = 1$ in λ and z , and the other one is of order $M = 1$ and independent of z . The latter polynomial, i.e., the third dimension, does not influence transport anymore because the system can be described in a spineless model with $M = 2$ and $m = 1$.

Performing the inverse counting statistics at $s = 1$ with the correct values $M = 2$ and $m = 1$ gives the spectrum depicted in figure 3(b). The input information is given

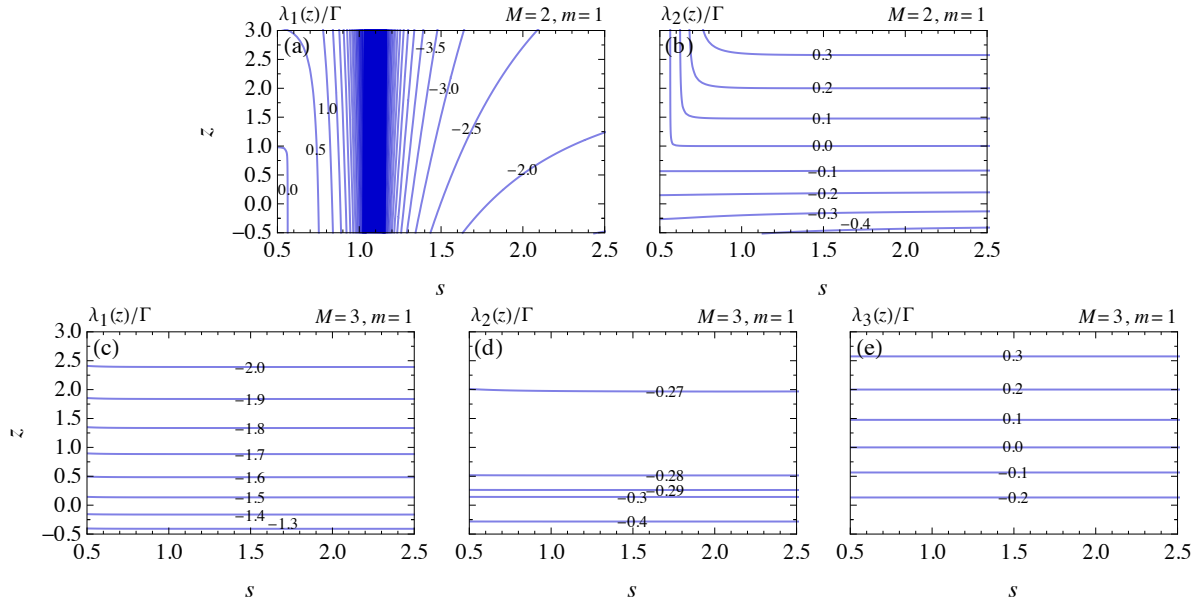


Figure 4. Consistency check of the dimension M and order m for the single-level quantum dot $\varepsilon = -k_B T$ in a magnetic field $\Delta = k_B T/2$ for time $\Gamma t = 6000$. Assuming different values for M and m , contour lines of the resulting $j = 1, \dots, M$ eigenvalues $\lambda_j(z)/\Gamma$ are depicted. The eigenvalues for $M = 2, m = 1$ in (a), (b) show a strong s -dependence, the ones for $M = 3, m = 1$ in (c), (d), (e) are s -independent.

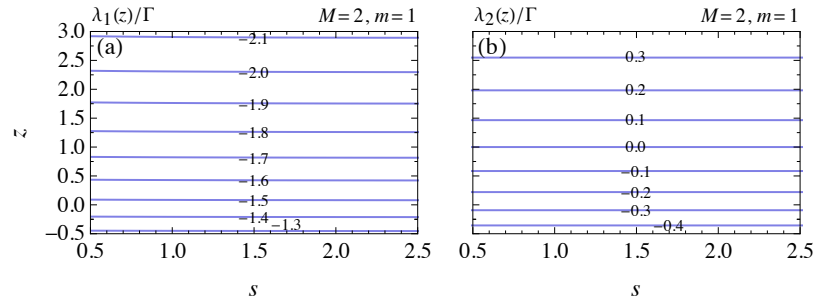


Figure 5. Consistency check of the dimension M and order m for the single-level quantum dot $\varepsilon = -k_B T$ without magnetic field for time $\Gamma t = 100$. Assuming $M = 2$ and $m = 1$, contour lines of the resulting two eigenvalues $\lambda_1(z)/\Gamma$ and $\lambda_2(z)/\Gamma$ are s -independent.

by the scaled long-time cumulants from order 0 to 3. For $\Gamma t = 100$, no difference to the exact long-time results can be recognized anymore.

For the consistency check of the dimensions M and m , we use as input information the calculated scaled cumulants at $\Gamma t = 100$. Horizontal contour lines in figure 5(a), (b) indicate that the eigenvalues are independent of s , i.e., the assumed dimensions $M = 2$ and $m = 1$ are compatible with the input data. In contrast for the choice $M = 3$ and $m = 1$, we obtain no s -independent spectrum of eigenvalues. The dimension $M = 3$ is too large.

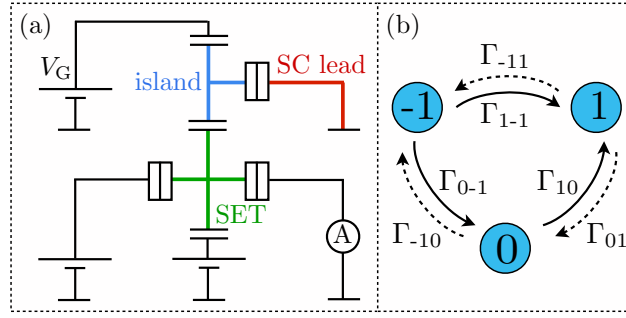


Figure 6. (Color online) (a) A normal-state metallic island (blue) weakly tunnel coupled to one superconducting lead (red). Via the gate voltage V_G , the gate charge n_G of the island can be tuned. By means of a current measurement through a single-electron transistor (green), the number of excess electrons n on the island is obtained as function of time. (b) Sketch of the states and transition rates. Dashed arrows indicate the counted transitions.

For completeness, we discuss in [Appendix B](#) how close the magnetic field has to be tuned to $\Delta = 0$ in order to observe the discussed behavior.

5. Sequential and Andreev tunneling in a single-electron box

The second example illustrating the inverse-counting statistics procedure is a model system that has been already experimentally realized in [\[62–64\]](#). The set up is depicted in figure 6(a). A single-electron box (SEB) is formed by one superconducting lead S weakly coupled (characterized by the tunnel resistance R_T) to a normal-state metallic island N . The energy required to bring n excess electrons on the island is $E_C(n - n_G)^2$. By applying a voltage V_G to a gate electrode, the gate charge is tuned near $n_G = 0$. The charge n on the island is monitored by an electrostatically coupled single-electron transistor (SET): each value of n results in a characteristic value of the current through the SET. Due to a finite temperature, transitions $n = 0 \rightarrow \pm 1$ with the rate Γ_{\pm}^{\pm} are possible (but temperature is small enough so that transitions to further charge states -2 and 2 are negligible). If the island is in one of these excited states, Andreev tunneling $n = \pm 1 \rightarrow \mp 1$ with the rate Γ_A^{\mp} is possible until the island relaxes back to the ground state $n = \pm 1 \rightarrow 0$ with rate Γ_d^{\mp} .

The stochastic system is depicted in figure 6(b). Its generator is given by

$$\mathbf{W}_z = \begin{pmatrix} -\Gamma_A^+ - \Gamma_d^+ & z\Gamma_u^- & z^2\Gamma_A^- \\ \Gamma_d^+ & -\Gamma_u^+ - \Gamma_u^- & z\Gamma_d^- \\ \Gamma_A^+ & \Gamma_u^+ & -\Gamma_A^- - \Gamma_d^- \end{pmatrix}. \quad (18)$$

The characteristic function is a polynomial of order $M = 3$ in λ and of order $m = 2$ in z . In general, the inverse counting statistics should deliver the z -dependence of all three eigenvalues.

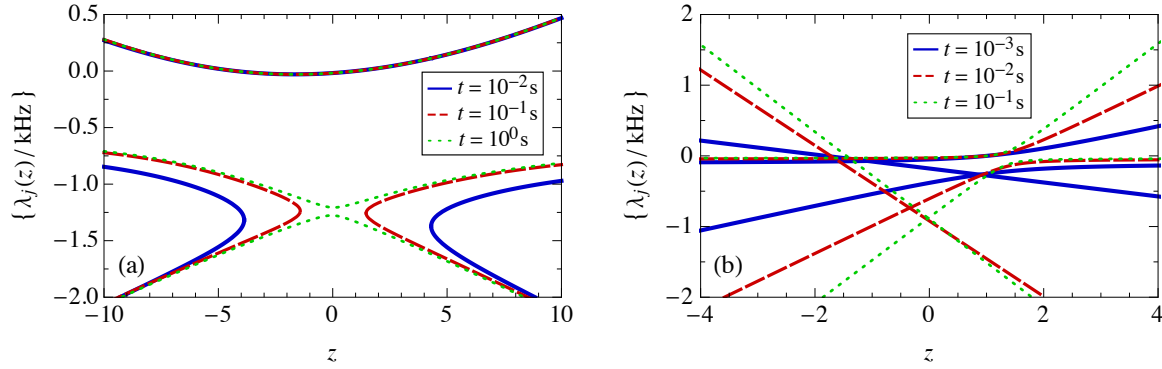


Figure 7. (Color online) Eigenvalue spectrum obtained via inverse counting statistics for a normal-state metallic island weakly tunnel coupled to one superconducting lead. The gate charge is (a) $n_G = 0.09$ and (b) $n_G = 0.00$. At the time (a) $t = 10^0$ s and (b) $t = 10^{-1}$ s no difference to the exact eigenvalues, obtained from equation (18), can be recognized anymore. Eigenvalues with a finite imaginary part are not depicted.

The symmetry point $n_G = 0$, however, is special. At this point, there are only three different rates $\Gamma_A = \Gamma_A^+ = \Gamma_A^-$, $\Gamma_d = \Gamma_d^+ = \Gamma_d^-$, and $\Gamma_u = \Gamma_u^+ = \Gamma_u^-$. As a consequence, the characteristic function becomes separable,

$$\chi(z, \lambda) = \chi_{1,2}(z, \lambda) \cdot \chi_3(z, \lambda) \quad (19)$$

$$\begin{aligned} \chi_{1,2}(z, \lambda) = & \lambda^2 + [(1-z)\Gamma_A + \Gamma_d + 2\Gamma_u]\lambda \\ & + 2(1-z)\Gamma_u(\Gamma_A + \Gamma_d) \end{aligned} \quad (20)$$

$$\chi_3(z, \lambda) = \lambda + (1+z)\Gamma_A + \Gamma_d. \quad (21)$$

The first factor is a polynomial of order $M = 2$ in λ and of order $m = 1$ in z , while the second one is of order $M = 1$ and $m = 1$. Depending on the choice of s , the inverse counting statistics will only provide the eigenvalue(s) of the first or the second factor. Away from the symmetry point, $n_G \neq 0$, the generator is not separable.

Instead of calculating the tunneling rates Γ_u^\pm , Γ_d^\pm , and Γ_A^\pm in the presence of an electromagnetic environment [62, 65], we rely on experimentally measured rates for $n_G = 0.09$ in [63] and $n_G = 0.00$ in [64]. In the former case, the experimental parameters are $n_G = 0.09$, $E_C = 43\mu\text{eV}$, $\Delta = 216\mu\text{eV}$, and $R_T = 2000\text{ k}\Omega$ at 60 mK temperature. The measured rates are $\Gamma_u^+ = 10.5\text{ Hz}$, $\Gamma_u^- = 7.2\text{ Hz}$, $\Gamma_d^+ = 1270\text{ Hz}$, $\Gamma_d^- = 730\text{ Hz}$, $\Gamma_A^+ = 460\text{ Hz}$, and $\Gamma_A^- = 23.0\text{ Hz}$. In the later case, the experimental parameters are $n_G = 0.00$, $E_C = 40\mu\text{eV}$, $\Delta = 210\mu\text{eV}$, and $R_T = 490\text{ k}\Omega$ at 50 mK temperature. The rates are $\Gamma_u^+ = \Gamma_u^- = \Gamma_u = 12\text{ Hz}$, $\Gamma_d^+ = \Gamma_d^- = \Gamma_d = 252\text{ Hz}$, and $\Gamma_A^+ = \Gamma_A^- = \Gamma_A = 615\text{ Hz}$.

5.1. Non-symmetric case

We start with discussing the generic, non-symmetric case, for which we choose the $n_G = 0.09$. The input information for the inverse counting statistics (for $M = 3$ and $m = 2$) is given by the scaled generalized factorial long-time cumulants from order 0 up

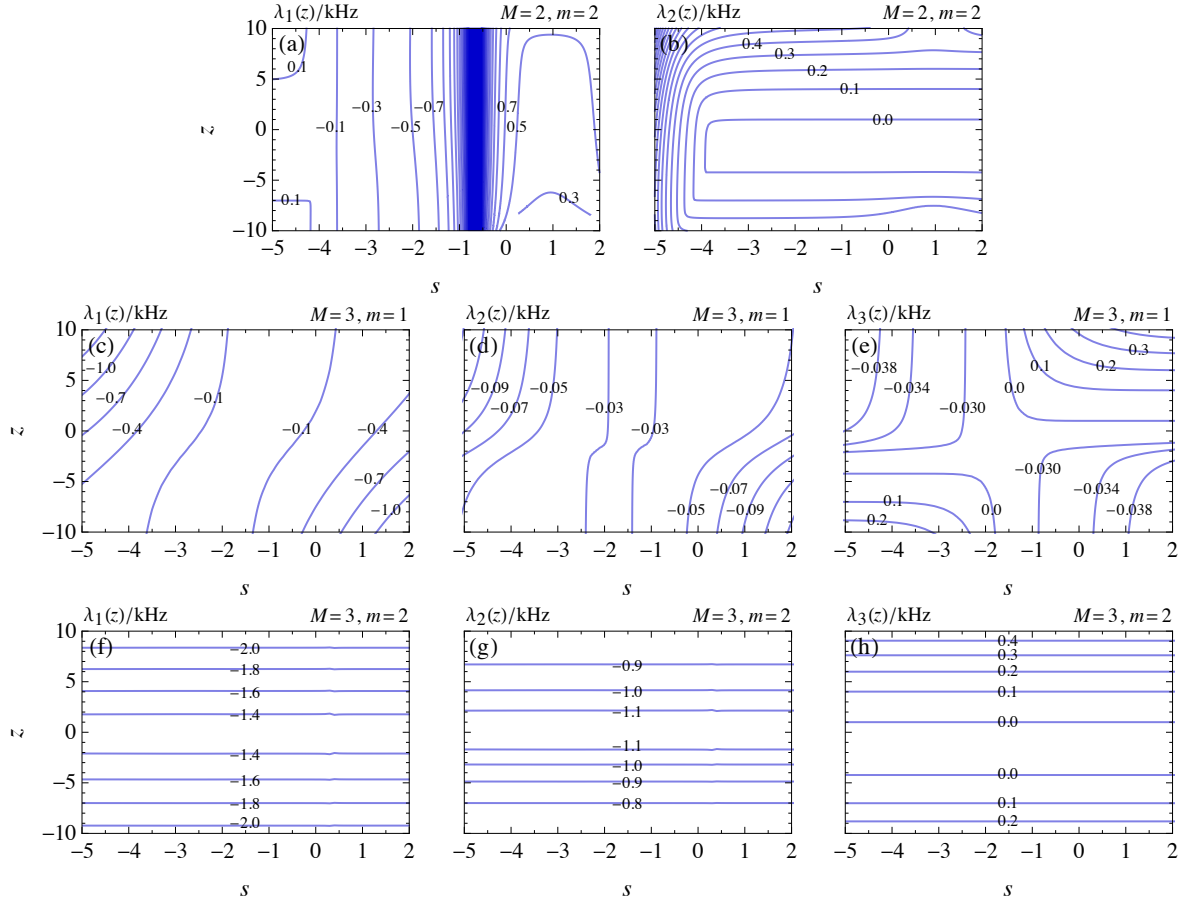


Figure 8. Consistency check of the dimension M and order m for the single-electron box with nonvanishing gate charge $n_G = 0.09$ for time $t = 10$ s. Assuming different values for M and m , contour lines of the resulting $j = 1, \dots, M$ eigenvalues $\lambda_j(z)/\text{kHz}$ are depicted. The eigenvalues for $M = 2, m = 2$ in (a), (b) are strongly s -dependent, for $M = 3, m = 1$ in (c), (d), (e) as well, but the eigenvalues for $M = 3, m = 2$ in (f), (g), (h) are s -independent.

to order $(m+1)M - 1 = 8$. Similar to the case of the single-level quantum dot discussed in section 4, we take into account that the measurement time is always finite in an experiment and do not take as input parameters the exact long-time scaled cumulants of the defined model but calculate, instead, the scaled cumulants at some large but finite time.

First, we determine the eigenvalue spectrum from the inverse counting statistics performed at $s = 1$ and assuming the correct values $M = 3$ and $m = 2$. As input parameter we take the calculated scaled generalized factorial cumulants at times $t = 0.01, 0.1, 1$ s. The result is shown in figure 7(a). For $t = 1$ s no difference to the exact long-time results can be recognized anymore.

Next, we demonstrate the consistency check for the dimensions M and m . For the right values, the eigenvalues must be s -independent. In the following, we always use as input information the calculated scaled cumulants at $t = 10$ s. To show simultaneously

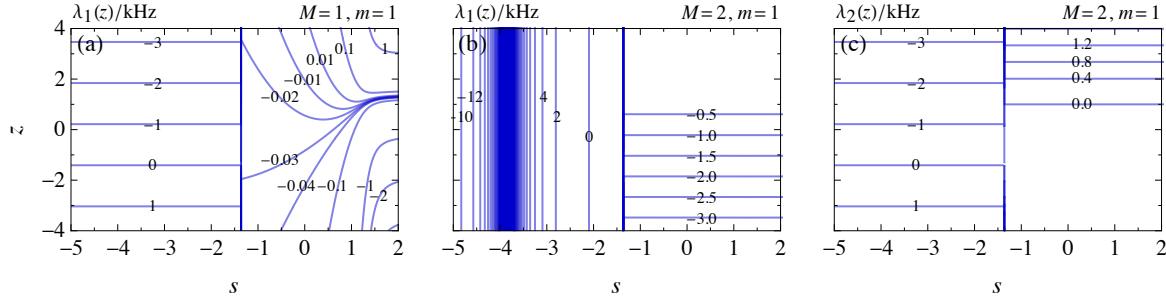


Figure 9. Consistency check of the dimension M and order m for the single-electron box with vanishing gate charge $n_G = 0$ for time $t = 20$ s. Assuming different values for M and m , contour lines of the resulting $j = 1, \dots, M$ eigenvalues $\lambda_j(z)/\text{kHz}$ are depicted. The eigenvalue for $M = 1, m = 1$ in (a) and the two eigenvalues for $M = 2, m = 1$ in (b), (c).

both the z - and the s -dependence of the eigenvalues, we plot in the following figures contour lines for λ (in units of kHz). Horizontal contour lines indicate that the eigenvalues are independent of s , i.e., the assumed dimensions M and m are compatible with the input data.

In figure 8(a), (b), we show the result for the choice $M = 2$ and $m = 2$. Since the dimension M of the stochastic systems is taken too small, the resulting eigenvalues display a strong s -dependence. The same holds true for the choice $M = 3$ and $m = 1$, see figure 8(c), (d), (e). In this case, the number m characterizing the coupling to the detector is taken too small, and, again, the resulting eigenvalues are s -dependent. Only if we take the proper values $M = 3$ and $m = 2$, see figure 8(f), (g), (h), we get s -independent results.

5.2. Symmetric case

We now turn to the symmetric case, $n_G = 0$, for which the characteristic function is separable, i.e., the characteristic function is a product of two polynomials, one of order $M = 2$ and $m = 1$ in λ and z , and the other one of order $M = 1$ and $m = 1$. The choice of s determines whether the eigenvalue with the largest real part is a root of the first or the second factor and, therefore, which and how many of the eigenvalues are accessible via the inverse counting statistics. If we choose $s = 1$ (corresponding to factorial cumulants), then we obtain only two of the three eigenvalues. For small $s < -1.36$ (for figure 7(b), we choose $s = -2$), we get only the third one. The number of required cumulants is 4 in the former and 2 in the latter case. The resulting spectrum of all three eigenvalues is depicted in figure 7(b) for the times $t = 0.001, 0.01, 0.1$ s. For $t = 0.1$ s, no difference to the exact long-time results can be recognized anymore.

The consistency check of the dimensions M and m (for $t = 20$ s) is depicted in figure 9(a) for $M = 1$ and $m = 1$ and in figure 9(b), (c) for $M = 2$ and $m = 1$. If we perform the inverse counting statistics around $s = 1$ (or for any $s > -1.36$), then

we conclude that the dimension $M = 1$ is too small (no horizontal contour lines in figure 9(a) for $s > -1.36$) but $M = 2$ seems to be sufficient (horizontal contour lines in Figs. 9(b) and (c) for $s > -1.36$). Thus, by employing only factorial cumulants ($s = 1$) one may be tempted to conclude that the dimension of the stochastic system is $M = 2$ only. If, on the other hand, inverse counting statistics is also done for $s < -1.36$, then the horizontal contour lines in figure 9(a) indicate that there is another eigenvalue. Since the obtained z -dependent eigenvalues are different from each other, we conclude that there must be, in total, three eigenvalues.

For completeness, we estimate in [Appendix C](#) how close n_G has to be tuned to zero in order to observe the discussed behavior.

6. Conclusions

In this paper, we propose inverse counting statistics based on generalized factorial cumulants as a convenient and powerful tool to reconstruct characteristic features of a stochastic system from measured counting statistics of some of the system's transitions. Such a method is particularly useful in cases in which very little is a priori known about the stochastic system under investigation. As the only input information for the inverse-counting-statistics procedure, we use a few experimentally determined numbers, namely the scaled generalized factorial cumulants in the long-time limit. Despite the limited amount of input, the inverse-counting-statistics procedure yields a remarkable extended amount of output. First, we can determine a lower bound of M , the dimension of the stochastic system. Second, we can find a lower bound of m , which characterizes the coupling between stochastic system and detector. Third, we can reconstruct the characteristic function $\chi(z, \lambda)$ of the generator \mathbf{W}_z , which is a polynomial of order M in λ and a polynomial of order m in z . From the zeros of the characteristic function, we can, then, determine the full z -dependence of the full spectrum of eigenvalues of \mathbf{W}_z . This is quite a remarkable result, since the long-time cumulants used as input depend only on one of the eigenvalues, λ_{\max} , determined around one value of z .

The use of generalized factorial cumulants instead of ordinary ones is crucial for our proposal. While the evaluation of generalized factorial cumulants from a measured time trace of the detector signal does not introduce any extra complication as compared to the evaluation of ordinary cumulants, the benefit of having a free parameter s in the definition of the generalized factorial cumulants is immense. First, the outcome of the inverse counting statistics must be s -independent. Therefore, an s -dependent outcome of the inverse counting statistics immediately indicates a wrong choice of M or m . Second, there are special cases of separable characteristic functions, for which the inverse counting statistics with ordinary cumulants would only reveal part of the spectrum of eigenvalues of the generator, while the variation of s makes it possible to access the full spectrum.

The proposed inverse counting statistics scheme is quite general and, therefore, applicable to large variety of systems. To illustrate the procedure we choose two

examples from electronic transport in nanostructures: a single-level quantum dot in a Zeeman field and a single-electron box subjected to sequential and Andreev tunneling. For the latter case, the full dimension of the system's state space and the full spectrum of eigenvalues can only be revealed by varying the parameter s .

Acknowledgments

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Appendix A. Alternative expression for $A_{l,\mu\nu}$

In the sum (13) identical terms, related by permutation of the indices $(\alpha_1, \dots, \alpha_\nu)$ may occur, e.g., $A_{\mu+2,\mu 2} = c_{s,0}c_{s,2}/2! + c_{s,1}c_{s,1} + c_{s,2}c_{s,0}/2! = c_{s,0}c_{s,2} + c_{s,1}^2$. If one wants to combine them, one obtains the alternative representation

$$A_{l,\mu\nu} = \sum_{\substack{\beta_0+\beta_1+\dots+\beta_\nu=\nu \\ \beta_1+2\beta_2+\dots+\nu\beta_\nu=l-\mu}} \frac{\nu!}{\beta_0! \cdot \beta_1! \cdots \beta_\nu!} \left(\frac{c_{s,0}}{0!}\right)^{\beta_0} \cdot \left(\frac{c_{s,1}}{1!}\right)^{\beta_1} \cdots \left(\frac{c_{s,\nu}}{\nu!}\right)^{\beta_\nu} \quad (\text{A.1})$$

Appendix B. Consistency check for $\Delta = 0.03 k_B T$

For completeness, we check how close the Zeeman field must be tuned to zero in order to observe the discussed behavior of section 4.2. We find that for $\Delta \lesssim 0.03 k_B T$, the eigenvalues obtained for $M = 2, m = 1$ are still almost s -independent (compare figure B1 to figure 5), with slight deviations appearing for very negative s .

Appendix C. Consistency check for $n_G = 0.001$

For the single-electron box, we estimate that at least for $|n_G| \lesssim 0.001$ the $n_G = 0$ case is already reached in good approximation for $s \lesssim -1.5$ or $s \gtrsim -0.5$ (compare

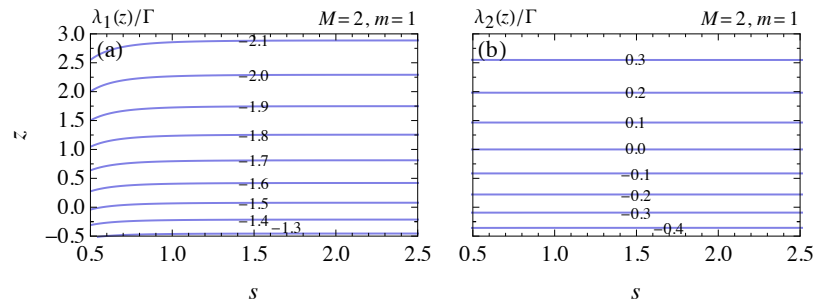


Figure B1. Consistency check of the dimension M and order m for the single-level quantum dot $\varepsilon = -k_B T$ in an external magnetic field $\Delta = 0.03 k_B T$ for time $\Gamma t = 100$. Assuming $M = 2$ and $m = 1$, contour lines of the resulting two eigenvalues $\lambda_1(z)/\Gamma$ and $\lambda_2(z)/\Gamma$ are depicted.

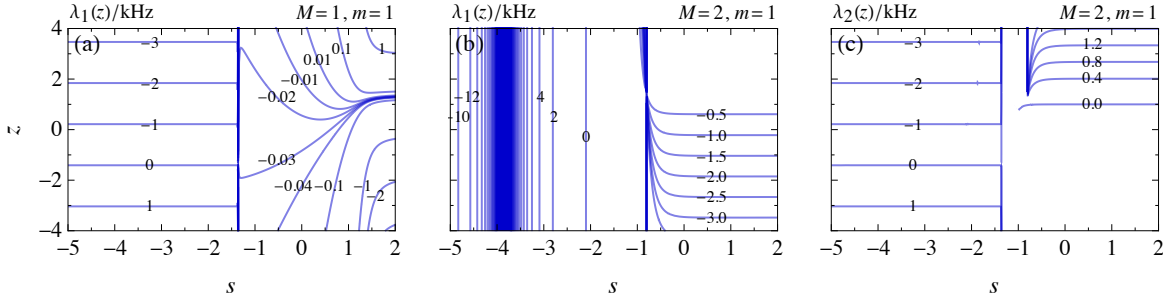


Figure C1. Consistency check of the dimension M and order m for the single-electron box with gate charge $n_G = 0.001$ for time $t = 20$ s. Assuming different values for M and m , contour lines of the resulting $j = 1, \dots, M$ eigenvalues $\lambda_j(z)/\text{kHz}$ are depicted. The eigenvalue for $M = 1, m = 1$ in (a) and the two eigenvalues for $M = 2, m = 1$ in (b), (c).

figure C1 to figure 9). For figure C1, the Andreev-tunneling rates are approximated by $\Gamma_A^\pm \approx (615.11 \pm 11.42)$ Hz [66]. The sequential tunneling rates are estimated via an interpolation between the experimental values of [63]: $\Gamma_u^\pm \approx (12.00 \pm 0.03)$ Hz and $\Gamma_d^\pm \approx (252.00 \pm 0.83)$ Hz.

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